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Recap

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$.
$\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ standard basis for $\mathbb{R}^n$.

Last time:

Th 3.4.3 Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transf. defined (in terms of $\mathcal{S}$) by an $n \times n$ matrix A.
Let $B = \left[[T(\vec{v}_1)]_{\mathcal{B}} \mid \cdots \mid [T(\vec{v}_n)]_{\mathcal{B}} \right]$. Then for any $x \in \mathbb{R}^n$,

$$[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$$

(Note B and $\mathcal{B}$ are totally different things)

(We know this when $\mathcal{B} = \mathcal{S}$.)

How are A and B related? This is an important topic.
We talked about the "change of basis" matrix S in the case of $\mathbb{R}^2$.
Here's the general case in a nutshell:

We have the standard basis $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$.
Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a different basis.

Let

$$\boxed{S = [\vec{v}_1 \mid \cdots \mid \vec{v}_n] \quad (n \times n \text{ invertible matrix})}$$

Let S^{-1} be its inverse.

For any vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ we have

$$S\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \left[\vec{v}_1 \mid \cdots \mid \vec{v}_n \right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n$$

Now, if $\vec{x} = a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n$ then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Hence

$$\boxed{\begin{aligned} S[\vec{x}]_{\mathcal{B}} &= \vec{x} = [\vec{x}]_{\mathcal{S}} \\ [\vec{x}]_{\mathcal{B}} &= S^{-1}[\vec{x}]_{\mathcal{S}} \end{aligned}}$$

So S takes us from the $\mathcal{B}$-coordinates to the $\mathcal{S}$-coordinates
S^{-1} " " " " $\mathcal{S}$ " " " $\mathcal{B}$ "

So if you know $\vec{x}$, you know $[\vec{x}]_{\mathcal{S}}$

$$\Rightarrow [\vec{x}]_{\mathcal{B}} = S^{-1}[\vec{x}]_{\mathcal{S}}$$

if you know $[\vec{x}]_{\mathcal{B}}$ then

$$\vec{x} = [x]_{\mathcal{S}} = S[\vec{x}]_{\mathcal{B}}.$$

It's the dictionary between the 2 bases!

Apply this to linear transformations.

Recall $B = \left[[T(\vec{v}_1)]_{\mathcal{B}} \mid \cdots \mid [T(\vec{v}_n)]_{\mathcal{B}} \right]$ and $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$

(Remember a matrix product corresp. to composition of functions so the one on the right goes first.)

Here's the key: (commutative diagram)

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \\ {[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"] \end{tikzcd}$$

in terms of $\mathcal{B}$

- Start with $\vec{x}$. To get $T(\vec{x})$ you either do $A\vec{x}$ or else you do

$$\vec{x} \xrightarrow{S^{-1}} [\vec{x}]_{\mathcal{B}} \longrightarrow B[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}} \longrightarrow S[T(\vec{x})]_{\mathcal{B}} = T(\vec{x})$$

So $\quad A = SBS^{-1}$ Note the order!

- Start with $[\vec{x}]_{\mathcal{B}}$. To get $[T(\vec{x})]_{\mathcal{B}}$ you either do $B[\vec{x}]_{\mathcal{B}}$ or else you do

$$[\vec{x}]_{\mathcal{B}} \xrightarrow{S} \vec{x} \xrightarrow{A} T(\vec{x}) \xrightarrow{S^{-1}} [T(\vec{x})]_{\mathcal{B}}$$

So $\quad B = S^{-1}AS$ Note the order!

<u>Theorem 3.4.4</u> With the above notation,

$$AS = SB$$

$$B = S^{-1}AS$$

$$A = SBS^{-1}$$

<u>Def</u> (3.4.5)

Two $n \times n$ matrices are <u>similar</u> if $\exists$ invertible matrix S such that

$$AS = SB, \text{ or equiv. } B = S^{-1}AS.$$

<u>Theorem</u> (3.4.6)

Similarity is an equivalence relation (reflexive, symmetric, transitive) leave the proof to you.

Now some examples like the HW.

<u>Ex</u> Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

Find the matrix B for T in terms of the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

<u>Sol'n</u>

Let $S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Check $S^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

↳ goes from $\mathcal{B}$ to $\mathcal{E}$

Then $B = S^{-1} A S$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -2 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -1 & -2 & -4 \\ 3 & 7 & 13 \end{bmatrix}$$

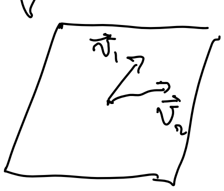
<u>Ex</u> Consider the plane V in $\mathbb{R}^3$ defined by $x_1 + 2x_2 + 3x_3 = 0$.

Find a basis $\mathcal{B}$ for V so that

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \text{where } \vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Note $\vec{x}$ is indeed a vector in V.

V



We want to find $\vec{v}_1, \vec{v}_2$ on V so that $\vec{x} = -2\vec{v}_1 + 4\vec{v}_2$

Idea: First pick anything nonzero on V for $\vec{v}_1$, say $\vec{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$

We want $\vec{v}_2 \in V$ so that

$$\vec{x} = -2\vec{v}_1 + 4\vec{v}_2$$

$$\Rightarrow \quad \vec{v}_2 = \frac{1}{4}\left(x + 2\vec{v}_1\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -10 \\ 2 \\ 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -11/4 \\ 1/4 \\ 3/4 \end{bmatrix}$$

Check: $-2\vec{v}_1 + 4\vec{v}_2 = \begin{bmatrix} 10 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} -11 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ ✓